The (unilateral) \( Z \)-transform of a sequence \( \{a_n\}_{n \geq 0} \) is defined as

\[
Z \{a_n\}_{n \geq 0} (z) = \sum_{n=0}^{\infty} a_n z^{-n}.
\]  

(1)

This definition is implemented in the Wolfram Language as `ZTransform[a, n, z]`. Similarly, the inverse \( Z \)-transform is implemented as `InverseZTransform[A, z, n]`.

"The" \( Z \)-transform generally refers to the unilateral \( Z \)-transform. Unfortunately, there are a number of other conventions. Bracewell (1999) uses the term "\( Z \)-transform" (with a lower case \( Z \)) to refer to the unilateral \( Z \)-transform. Girling (1987, p. 425) defines the transform in terms of samples of a continuous function. Worse yet, some authors define the \( Z \)-transform as the bilateral \( Z \)-transform.

In general, the inverse \( Z \)-transform of a sequence is not unique unless its region of convergence is specified (Zwillinger 1996, p. 545). If the \( Z \)-transform of a function is known analytically, the inverse \( Z \)-transform can be computed using the contour integral

\[
\alpha_k = \frac{1}{2\pi i} \int_{C} \overline{F}(\xi) e^{zk} \, d\xi,
\]

(2)

where \( C \) is a closed contour surrounding the origin of the complex plane in the domain of analyticity of \( F(z) \) (Zwillinger 1996, p. 545).

The unilateral transform is important in many applications because the generating function \( G(z) \) of a sequence of numbers \( \{a_n\}_{n \geq 0} \) is given precisely by \( Z \{a_n\}_{n \geq 0} (z) \) in the variable \( 1/z \) (Germundsson 2000). In other words, the \( Z \)-transform of a function \( f(t) \) gives precisely the sequence of terms in the series expansion of \( f(t) \). So, for example, the terms of the series of \( z^{-1} \left( t^{-1} - 1 \right) / z^{-1} \) are given by

\[
Z \{ t^{-1} \left( t^{-1} - 1 \right) / z^{-1} \} = \frac{z^{-1} (z^{-1} - 1)}{z^{-1}} = z^{-1} - 1.
\]

Girling (1987) defines a variant of the unilateral \( Z \)-transform that operates on a continuous function \( F(t) \) sampled at regular intervals \( T \),

\[
Z_T \{ F(t) \} (z) = L \{ F(t) \} (z/T),
\]

(4)

where \( L \{ f(t) \} \) is the Laplace transform,

\[
L \{ f(t) \} (s) = \int_{0}^{\infty} f(t) e^{-st} \, dt
\]

(5)

and the one-sided shah function with period \( T \) is given by

\[
\delta_T (t) = \sum_{n=-\infty}^{\infty} \delta (t - nT),
\]

(6)

where

\[
\delta (t) = \begin{cases} \delta_{\text{uw}}(t), & t \geq 0, \\ 0, & t < 0, \end{cases}
\]

(7)

and \( \delta_{\text{uw}} \) is the Kronecker delta, giving

\[
Z_T \{ F(t) \} (z) = \sum_{n=-\infty}^{\infty} F(nT) z^{-n}.
\]

(8)

An alternative equivalent definition is

\[
Z_T \{ F(t) \} (z) = \sum_{n=-\infty}^{\infty} \frac{1}{t - nT} F(t) z^{-n},
\]

(9)

where

\[
F(z) = \sum_{n=-\infty}^{\infty} F(nT) z^{-n}.
\]

(10)

This definition is essentially equivalent to the usual one by taking \( a_n = F(nT) \).

The following table summarizes the \( Z \)-transforms for some common functions (Girling 1987, pp. 426-427; Bracewell 1999). Here, \( \delta_{\text{uw}} \) is the Kronecker delta, \( H(t) \) is the Heaviside step function, and \( \text{Li}_m(z) \) is the polylogarithm.
The $Z$-transform of the general power function $x^k$ can be computed analytically as

$$Z \left( \frac{1}{k!} \right) = \left( \frac{1}{k!} \right) \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{k!} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{k!} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{k!} \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(11)

(12)

(13)

where the $\binom{n}{k}$ are Eulerian numbers and $Li_n(q)$ is a polylogarithm. Amazingly, the $Z$-transforms of $x^k$ are therefore generators for Euler's number triangle.

The $Z$-transform $Z \left\{ \text{[a] } [b] \right\} = \{c\}$ satisfies a number of important properties, including linearity

$$Z \left\{ a[n] + b[n] \right\} = a Z \left\{ [a] \right\} + b Z \left\{ [b] \right\}$$

(14)

translation

$$Z \left\{ e^{\alpha n} a[n] \right\} = F \left( \frac{\alpha}{b} \right)$$

(15)

(16)

(17)

(18)

and multiplication by powers of $x$

$$Z \left\{ x^n a[n] \right\} = \frac{d^n}{dz^n} \left( F \left( \frac{z}{b} \right) \right)$$

(19)

(20)

(21)


The discrete Fourier transform is a special case of the $Z$-transform with

$$Z \left\{ e^{2\pi i \theta n} a[n] \right\} = F \left( e^{-2\pi i \theta / b} \right)$$

(22)

and a $Z$-transform with

$$Z \left\{ e^{2\pi i \theta n} a[n] \right\} = F \left( e^{-2\pi i \theta / b} \right)$$

(23)

for $\theta \in [-\pi, \pi]$ is called a fractional Fourier transform.

SEE ALSO:
Bilateral Z-Transform, Discrete Fourier Transform, Euler's Number Triangle, Eulerian Number, Fractional Fourier Transform, Generating Function, Laplace Transform, Population Comparison, Unilateral Z-Transform

REFERENCES:
Arndt, J. "The $Z$ Transform (ZT)." Ch. 3 in Remarks on FFT Algorithms. http://www.jjj.de/fxt/.
In mathematics and signal processing, the Z-transform converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency-domain representation. It can be considered as a discrete-time equivalent of the Laplace transform. This similarity is explored in the theory of time-scale calculus. The basic idea now known as the Z-transform was known to Laplace, and it was re-introduced in 1947 by W. Hurewicz and others as a way to treat sampled-data control systems used